Benha University Faculty of Engineering- Shoubra Eng. Mathematics & Physics Department Preparatory Year		Final Term Exam Date: 28 – 12 – 2014 Course: Mathematics 1 – A Duration: **3** hours

- Answer **All** questions
- The Exam Consists of One page
- No. of questions: 4
- Total Mark: 100 Marks

[1] Find **y`** from the following: **24**

(a) $y = 3x^{-4} + 3^{\sin x} + \frac{1}{2}$ (b) $y = \cosh 2x \, . \sinh x^2$ (c) $y = \tan x \, . \ln(2^x + \sqrt{3})$

(d) $y = \tanh^{-1} x^2 + \sin^{-1} x$ (e) $y = x^{\sqrt{x}} + (\sinh x)^x$ (f) $y \cos x + x \sin y = 2$

(g) $y = \dfrac{\sqrt{\sin^5 x + \sin x^5}}{\sqrt[8]{x + \cos x} . \sqrt[4]{x + \cosh x}}$ (h) $y = t^2 + \ln \cos t, \quad x = t + \tan^{-1} \ln t$

[2](a)Find the following limits: **9**

(i) $\lim\limits_{x \to 0} \dfrac{x^4 + \sin^4 x}{x^5 + \tan^4 x}$ (ii) $\lim\limits_{x \to \infty} (x - \ln(1 + e^x))$ (iii) $\lim\limits_{x \to \frac{\pi}{2}} (\sec x - \tan x)$

(b)Write the Maclurin's series of the function: $f(x) = x \ln(2x + 1)$ **5**

(c)Determine the maximum and minimum points of the functions: **12**

$$f(x) = x . e^{-x}, \quad g(x) = 1 - x^{\frac{2}{3}}, \quad h(x) = x^2 - 4 \ln(x + 1).$$

[3](a)Find the integrals: (i) $\displaystyle\int \frac{x^2 dx}{(4 - x^2)^{3/2}}$ (ii) $\displaystyle\int x^2 \tan^{-1} x \, dx$ (iii) $\displaystyle\int \cos^3 x \sin^{-4} x dx$ **15**

(b) Find the area bounded by: $r = \dfrac{6}{1 + \cos\theta}$ and y-axis. **5**

(c) Find the circumference of cardioid: $r = a(1 - \cos\theta)$ **5**

[4](a)Find the integrals: (i) $\displaystyle\int \frac{1}{x^{5/2}(3x + 1)^{3/2}} dx$ (ii) $\displaystyle\int \frac{\sin x \, dx}{\cos x (1 + \cos^2 x)}$ **10**

(b)If $I_m = \int \tan^m x \, dx$ show that $I_m = \dfrac{\tan^{m-1} x}{m - 1} - I_{m-2}$ **5**

(c)Find the volume generated by revolving the cycloid: $x = \theta - \sin\theta, \quad y = 1 - \cos\theta$ **5**

about y – axis.

(d)Find the area of the surface of revaluation generated by revolving the hypocycloid **5**

$x = a \cos^3\theta, \quad y = a \sin^3\theta$ about ***x***-axis.

Good Luck *Dr. Mohamed Eid* *Dr. Fathi Abdsallam*

Model Answer

Answer of Question (1): 24 Marks

(a) $y` = -12x^{-5} - 3^{\sin x}.\ln 3 . \cos x$

(b) $y` = \cosh 2x . \cosh x^2 . 2x + 2 \sinh 2x . \sinh x^2$

(c) $y = \tan x . \frac{2^x \ln 2}{2^x + \sqrt{3}} + sec^2 x . \ln(2^x + \sqrt{3})$

(d) $y` = \frac{2x}{1-x^4} + \frac{1}{\sqrt{1-x^2}}$

(e) $y` = e^{\sqrt{x} ln\, x}\left(\frac{1}{2\sqrt{x}} \ln x + \frac{\sqrt{x}}{x}\right) +$

$e^{x\, ln \sinh x}\left(\ln \sinh x + x\frac{\cosh x}{\sinh x}\right)$

(f) $y` \cos x - y \sin x + \sin y + x \cos y . y` = 0$

Then $y` = \frac{y \sin x - \sin y}{\cos x + x \cos y}$

(g) $\ln y = \frac{1}{2}\ln(\sin^5 x + \sin x^5) -$

$\frac{1}{8}\ln(x + \cos x) - \frac{1}{4}\ln(x + \cosh x)$

(h) $y` = \frac{2t + \frac{-\sin t}{\cos t}}{1 + \frac{1/t}{1+(\ln t)^2}}$

Then $\frac{y`}{y} = \frac{1}{2}\frac{5\sin^4 x . \cos x + \cos x^5 . 5x^4}{\sin^5 x + \sin x^5} -$

$\frac{1}{8}\frac{1 - \sin x}{x + \cos x} - \frac{1}{4}\frac{1 + \sinh x}{x + \cosh x}$

Answer of Question (2): 26 Marks

(a)(i) $\text{Lim}_{\ x\to 0} \frac{x^4 + \sin^4 x}{x^5 + \tan^4 x} = \frac{0}{0} = \text{Lim}_{\ x\to 0} \frac{1 + \frac{\sin^4 x}{x^4}}{x + \frac{\tan^4 x}{x^4}} = \frac{1+1}{0+1} = 2$

(ii) $\text{Lim}_{x\to\infty}(x - \ln(1 + e^x)) = \infty - \infty = \text{Lim}_{x\to\infty} \ln[\frac{e^x}{1+e^x}] = \text{Lim}_{x\to\infty} \ln[\frac{e^x}{e^x}] = \ln 1 = 0$

(iii) $\text{Lim}_{x\to\frac{\pi}{2}}(\sec x - \tan x) = \infty - \infty = \text{Lim}_{x\to\frac{\pi}{2}} \frac{1 - \sin x}{\cos x} = \text{Lim}_{x\to\frac{\pi}{2}} \frac{\cos x}{-\sin x} = 0$

(b) From $f`(x) = e^{-x} - xe^{-x} = 0$, then x = 1.

Since $f``(x) = -2e^{-x} + xe^{-x}$, $f``(1) = -e^{-1}$. Then the point x = 1 is maximum.

From $g`(x) = -\frac{2}{3}x^{-\frac{1}{3}} \neq 0$ and $g`(0) = \infty$ but f(0) = 1. Then we test this point.

Since $g`(0^-) = g`(-1) = \frac{2}{3}$ *and* $g`(0^+) = g`(1) = -\frac{2}{3}$

Then the point x = 0 is maximum.

From $h`(x) = 2x - \frac{4}{x+1} = 0$. Then $x^2 + x - 2 = 0$. Then x = – 2, x = 1.

Since $g``(x) = 2 + 4(x+1)^{-2}$ *and* $g``(1) = 3$. Then the point x = 1 is minimum.

The point x = – 2 is not critical because h(–2) is not real.

(c)If $y = \operatorname{sech}^{-1} x$. Then $x = \operatorname{sech} y = \frac{2}{e^y + e^{-y}}$. Then $x = \frac{2e^y}{e^{2y}+1}$.

Then $xe^{2y} - 2e^y + x = 0$. Then $e^y = \frac{1}{2x}[2 \pm \sqrt{4-4x^2}] = \frac{1}{x}[1 \pm \sqrt{1-x^2}]$

Then $y = \ln\left[\frac{1+\sqrt{1-x^2}}{x}\right] = \ln\left[1+\sqrt{1-x^2}\right] - \ln x$

(d) Since $f(0) = 0$,

$$f`(x) = \ln(1+2x) + \frac{2x}{1+2x} = \ln(1+2x) + 1 - \frac{1}{1+2x} \quad and \quad f`(0) = 0$$

$$f``(x) = \frac{2}{1+2x} + 0 + \frac{2}{(1+2x)^2} \quad and \quad f``(0) = 4$$

$$f```(x) = -4(1+2x)^{-2} - 8(1+2x)^{-3} \quad and \quad f```(0) = -12$$

$$f^{(4)}(x) = 16(1+2x)^{-3} + 48(1+2x)^{-4} \quad and \quad f^{(4)}(0) = 64$$

Then $f(x) = 0 + 0x + \frac{4}{2!}x^2 - \frac{12}{3!}x^3 + \frac{64}{4!}x^4 + \cdots$

Dr. Mohamed Eid

<u>Answer of Question (3): 25 Marks</u>

(i) $\int \frac{x^2 dx}{(4-x^2)^{3/2}}$ Use trigonometric substitution put $x = 2\sin\theta$

$\therefore dx = 2\cos\theta d\theta$ and $x^2 = 4\sin^2\theta$, $\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = 2\cos\theta$

$$\int \frac{x^2 dx}{(4-x^2)^{3/2}} = \int \frac{(4\sin^2\theta).(2\cos\theta d\theta)}{(4-4\sin^2\theta)^{3/2}} = \int \frac{(4\sin^2\theta.)(2\cos\theta d\theta)}{(4\cos^2\theta)^{3/2}} = \frac{8}{8}\int \frac{4\sin^2\theta\cos\theta d\theta}{\cos^3\theta}$$

$$= \int \frac{\sin^2\theta d\theta}{\cos^2\theta} = \int \tan^2\theta d\theta \int (\sec^2\theta - 1)d\theta = \tan\theta - \theta + c = \boxed{\frac{x}{\sqrt{4-x^2}} - \sin^{-1}\frac{x}{2} + c}$$

(ii) $\int x^2 \tan^{-1} x\, dx$ Integrate by parts

$u = \tan^{-1} x \qquad dv = x^2 dx$

$\therefore du = \frac{dx}{1+x^2} \qquad let \ \ v = \frac{1}{3}x^3$

$$\therefore \int u\,dv = \int x^2 \tan^{-1} x\,dx = \frac{1}{3}x^3 \tan^{-1} x - \frac{1}{3}\int \frac{x^3\,dx}{1+x^2} = \frac{1}{3}x^3 \tan^{-1} x - \frac{1}{3}\int (x - \frac{x}{1+x^2})dx$$

$$= \frac{1}{3}x^3 \tan^{-1} x - \frac{1}{3}\left(\frac{1}{2}x^2 - \ln(1+x^2)\right) + c = \boxed{\frac{1}{3}x^3 \tan^{-1} x - \frac{1}{6}x^2 + \frac{1}{6}\ln(1+x^2) + c}$$

(iii) $\int \cos^3 x \sin^{-4} x dx = \int \cos^2 x \sin^{-4} x(\cos x dx) = \int (1-\sin^2 x)\sin^{-4} x(\cos x dx)$

$$= \int ((\sin x)^{-4} - (\sin x)^{-2})(\cos x dx) = \frac{-1}{3}(\sin x)^{-3} + (\sin x)^{-1} + c$$

$$= \boxed{\frac{-1}{3}\csc^3 x + \csc x + c}$$

(b) $\boldsymbol{r = \frac{6}{1+\cos\theta}}$ replace $\boldsymbol{\theta}$ by $\boldsymbol{-\theta}$ we find that r does not change which indicate that the curve symmetrical about x- axis then the required area given by

$$A = 2\left(\frac{1}{2}\int_0^{\pi/2} r^2 d\theta\right) = \int_0^{\pi/2} \frac{36}{(1+\cos\theta)^2} d\theta = \int_0^{\pi/2} \frac{36}{(2\cos^2 \frac{\theta}{2})^2} d\theta$$

$$= \frac{36}{4}\int_0^{\pi/2} \sec^4 \frac{\theta}{2} d\theta = 9\int_0^{\pi/2} (\sec^2 \frac{\theta}{2})(\sec^2 \frac{\theta}{2}) d\theta = 9\int_0^{\pi/2} (\sec^2 \frac{\theta}{2})(1+\tan^2 \frac{\theta}{2}) d\theta$$

$$= 9\int_0^{\pi/2} (\sec^2 \frac{\theta}{2} + \sec^2 \frac{\theta}{2}\tan^2 \frac{\theta}{2}) d\theta = 9\left[2\tan\frac{\theta}{2} + \frac{2}{3}\tan^3 \frac{\theta}{2}\right]_0^{\pi/2}$$

$$= 9\left[2\tan\frac{\pi}{4} + \frac{2}{3}\tan^3 4\right] = 9\left[2 + \frac{2}{3}\right] = \boxed{24 \text{ square unit}}$$

<u>Another solution</u>

$\boldsymbol{r = \frac{6}{1+\cos\theta}}$ transfer the equation to the Cartesian form as following

$r = \frac{6}{1+\cos\theta} \Rightarrow r + r\cos\theta = 6$ substitute by $x = r\cos\theta$ *then* $r = 6 - x$ and

$$r^2 = 36 - 12x + x^2 \quad \text{since } r^2 = x^2 + y^2$$

$$\therefore x^2 + y^2 = 36 - 12x + x^2 \quad \Rightarrow \boxed{y^2 = 36 - 12x}$$

Which represent a parabola with intercept x-axis at x=3 and y axis at **(0,$\pm$ 6)**

Required aria given by

$$A = 2\int_0^3 y dx = 2\int_0^3 (36-12x)^{1/2} dx = 2(\frac{2}{3})(\frac{-1}{12})\left[(36-12x)^{3/2}\right]_0^3 = 2(\frac{2}{3})(\frac{-1}{12})(-6)^3 = 24 \; unit$$

Another solution

The parametric equation is

$$x = r\cos\theta = \frac{6\cos\theta}{1+\cos\theta} \quad and \quad y = r\sin\theta = \frac{6\sin\theta}{1+\cos\theta}$$

$$x = r\cos\theta = \frac{6\cos\theta}{1+\cos\theta} = \frac{6(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2})}{2\cos^2\frac{\theta}{2}} = 3(1-\tan^2\frac{\theta}{2})$$

$$y = r\sin\theta = \frac{6\sin\theta}{1+\cos\theta} = \frac{6(2\sin\frac{\theta}{2}\cos\frac{\theta}{2})}{2\cos^2\frac{\theta}{2}} = 6\tan\frac{\theta}{2}$$

put $\boldsymbol{t = \tan\frac{\theta}{2}}$ then $\quad x = 3(1-t^2) \;\; and \;\; y = 6\text{t}$ The curve bisect y axis at y=1 and y=-1

$$A = 2\int_0^1 y dx = 2\int_0^1 6t(-6t)dx = 2(36)\left(\frac{t^3}{3}\right)_0^1 = 24$$

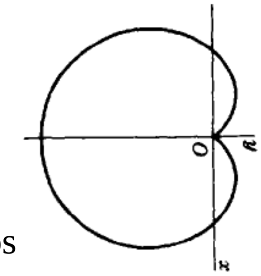
Also we can construct the Cartesian equation

$$x = 3(1-\frac{1}{36}y^2) = 3 - \frac{1}{12}y^2 \qquad \Rightarrow y^2 = 36 - 12x$$

(c) $\boldsymbol{r = a(1-\cos\theta)}$ $\qquad \therefore \boldsymbol{\frac{dr}{d\theta} = a\sin\theta}$

$$r^2+\left[\frac{dr}{d\theta}\right]^2=a^2(1-\cos\theta)^2+a^2\sin^2\theta=a^2\frac{1}{2}(2-2\cos\theta)$$

$$=2a^2(1-\cos\theta)=4a^2\sin^2\frac{\theta}{2}$$



$$L=\int_0^{2\pi}\sqrt{r^2+\left[\frac{dr}{d\theta}\right]^2}=\int_0^{2\pi}\sqrt{4a^2\sin^2\frac{\theta}{2}}\,d\theta=2a\int_0^{2\pi}\sin\frac{\theta}{2}\,d\theta=4a\Big[-\cos$$

<u>Answer of Question (4): 25 Marks</u>

(a) **(i)** $\int\frac{1}{x^{5/2}(3x+1)^{3/2}}\,dx=\int\frac{1}{x^{5/2}x^{3/2}(\frac{3x+1}{x})^{3/2}}\,dx=\int\frac{1}{x^4(\frac{3x+1}{x})^{3/2}}\,dx$

Put $u=\frac{3x+1}{x}$ then $u=3+\frac{1}{x}$ and $x=(u-3)^{-1}$ by differentiate we have

$dx=-(u-3)^{-2}du$ Substitute in the problem

$$\int\frac{1}{x^4(\frac{3x+1}{x})^{3/2}}\,dx=\int\frac{(u-3)^{-2}du}{(u-3)^{-4}(u)^{3/2}}=\int\frac{(u-3)^2du}{(u)^{3/2}}=\int\frac{(u^2-6u+9)}{(u)^{3/2}}du$$

$$=\int\left(u^{1/2}-6u^{-1/2}+9u^{-3/2}\right)du=\left(\frac{2}{3}u^{3/2}-12u^{1/2}-18u^{-1/2}\right)$$

$$=\boxed{\frac{2}{3}\left(\frac{3x+1}{x}\right)^{3/2}-12\left(\frac{3x+1}{x}\right)^{1/2}-18\left(\frac{3x+1}{x}\right)^{-1/2}+C}$$

(ii) $\int\frac{\sin x\,dx}{\cos x(1+\cos^2 x)}$

Put $u=\cos x$ then $du=-\sin x dx$ then becomes $\int\frac{\sin x\,dx}{\cos x(1+\cos^2 x)}=-\int\frac{du}{u(1+u^2)}$

Resolve the fraction $\boldsymbol{\frac{1}{u(1+u^2)}}$ into its partial fractions $\frac{1}{u(1+u^2)}=\frac{A}{u}+\frac{Bu+C}{(1+u^2)}$

$1=A(1+u^2)+u(Bu+C)\quad then\ A=1,\ B=-1, C=0\ and\ \frac{1}{u(1+u^2)}=\frac{1}{u}-\frac{u}{(1+u^2)}$

$$-\int\frac{du}{u(1+u^2)}=-\int\frac{du}{u}+\int\frac{udu}{(1+u^2)}=-\ln u+\frac{1}{2}\ln(1+u^2)=-\ln u+\ln\sqrt{1+u^2}=\ln\frac{\sqrt{1+u^2}}{u}+c$$

$$\int\frac{\sin x\,dx}{\cos x(1+\cos^2 x)}=\boxed{\ln\frac{\sqrt{1+\cos^2 x}}{\cos x}+c}$$

(b) $I_m=\int\tan^m x\,dx=\int\tan^{m-2}x\tan^2 x\,dx=\int\tan^{m-2}x(\sec^2 x-1)\,dx$

$$=\int\left(\tan^{m-2}x\sec^2 x-\tan^{m-2}x\right)dx=\frac{\tan^{m-1}x}{m-1}-\int\tan^{m-2}x dx=\frac{\tan^{m-1}x}{m-1}-I_{m-2}$$

(c) $V=2\pi\int_{t=0}^{t=2\pi}xydx=2\pi\int_{t=0}^{t=2\pi}(\theta-\sin\theta)(1-\cos\theta)(1-\cos\theta)d\theta$

$$=2\pi\int_{t=0}^{t=2\pi}(\theta-2\theta\cos\theta+\theta\cos^2\theta-\sin\theta+2\sin\theta\cos\theta-\cos^2\theta\sin\theta)d\theta$$

$$=2\pi\left[\frac{3}{4}\theta^2-2(\theta\sin\theta+\cos\theta)+\frac{1}{2}(\frac{1}{2}\theta\sin 2\theta+\frac{1}{4}\cos 2\theta)+\cos\theta+\sin^2\theta+\frac{1}{3}\cos^3\theta\right]_0^{2\pi}=\boxed{6\pi^3}$$

(d) We shall use $S_x=2\pi\int_a^b y\sqrt{\left[\frac{dx}{d\theta}\right]^2+\left[\frac{dy}{d\theta}\right]^2}\ d\theta$

since $x=a\cos^3\theta,\ y=a\sin^3\theta$ then $\frac{dx}{d\theta}=-3a\cos^2\theta\sin\theta,\quad \frac{dy}{d\theta}=3a\sin^2\theta\cos\theta$

$$\left[\frac{dx}{d\theta}\right]^2+\left[\frac{dy}{d\theta}\right]^2=9a^2\cos^4\theta\sin^2\theta+9a^2\sin^4\theta\cos^2\theta=9a^2\sin^2\theta\cos^2\theta$$

$$y\sqrt{\left[\frac{dx}{d\theta}\right]^2+\left[\frac{dy}{d\theta}\right]^2}=a\sin^3\theta\sqrt{9a^2\sin^2\theta\cos^2\theta}=3a^2\sin^4\theta\cos\theta$$

$$\therefore S_x=2\pi\int_a^b y\sqrt{\left[\frac{dx}{d\theta}\right]^2+\left[\frac{dy}{d\theta}\right]^2}\,d\theta=2(2\pi)\int_0^{\pi/2} 3a^2\sin^4\theta\cos\theta d\theta$$

$$=\frac{12\pi a^2}{5}\left[\sin 5\theta\right]_0^{\pi/2}=\boxed{\frac{12\pi a^2}{5}}$$

Dr. Fathi Abdessalam